APPROXIMATE SOLUTION OF THE LAMINAR BOUNDARY LAYER EQUATION FOR A NON-NEWTONIAN FLUID ON A PLATE

K. I. Strakhovich and S. L. Lur'e

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The laminar boundary layer equations for a non-Newtonian fluid on a flat plate are reduced to Prandtl-Mises variables and solved approximately in quadratures. The velocity profiles and the resistance coefficients are given for certain values of n. Agreement with the results of exact calculations is good.

For a fluid satisfying the rheological power law

$$\tau = K\left(\frac{\partial u}{\partial y}\right)^n, \tag{1}$$

the laminar boundary layer equations on a flat plate, in dimensionless form, are [1]

$$u_1\frac{\partial u_1}{\partial x_1} + v_1\frac{\partial u_1}{\partial y_1} = \frac{\partial}{\partial y_1}\left(\frac{\partial u_1}{\partial y_1}\right)^n, \quad \frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} = 0, \tag{2}$$

where

$$x_1 = \frac{x}{L}, \quad y_1 = \frac{y}{L}R^{\frac{1}{1+n}},$$

$$u_1 = \frac{u}{U}, \quad v_1 = \frac{v}{U}R^{\frac{1}{1+n}},$$

$$R = \frac{\rho U^{2-n}L^n}{K}. \tag{3}$$

The boundary conditions are

$$\text{when } y_1 = 0 \quad u_1 = 0, \quad v_1 = 0;$$
$$\text{when } y_1 = \infty \quad u_1 = 1. \tag{4}$$

Introducing the stream function ψ_1, we replace the second equation of (2) by the relations

$$u_1 = \frac{\partial \psi_1}{\partial y_1}, \quad v_1 = -\frac{\partial \psi_1}{\partial x_1}. \tag{5}$$

We shall pass from x_1, y_1 to the new independent variables x_1, $\psi_1(x_1, y_1)$—Prandtl-Mises variables [2], and transform the first equation of (2) into these variables. Thus, it now takes the form

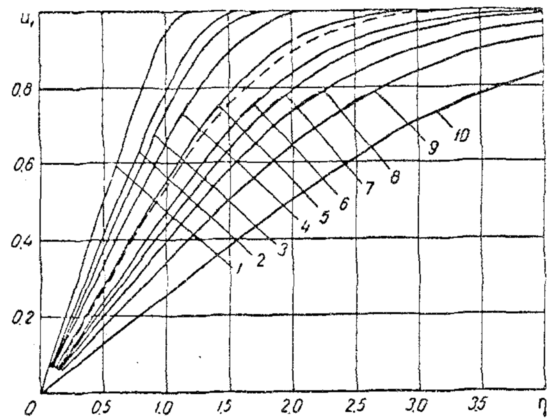
$$\frac{\partial z}{\partial x_1} = \sqrt{2z}\frac{\partial}{\partial \psi_1}\left(\frac{\partial z}{\partial \psi_1}\right)^n, \tag{6}$$

where we have designated

$$z = u_1^2/2. \tag{7}$$

Taking the straight line $y_1 = 0$ as the zero stream line ($\psi_1 = 0$), we write the boundary conditions (4), taking (7) into account, as

$$\text{when } \psi_1 = 0 \quad z = 0;$$
$$\text{when } \psi_1 = \infty \quad z = 1/2. \tag{8}$$



Velocity distributions in the boundary layer with: 1) n = 3; 2) 2; 3) 1.67; 4) 1.33; 5) 1.0; 6) 0.7; 7) 0.7; 8) 0.6; 9) 0.5; and 10) 0.3.

If z is regarded as being a function of only one variable

$$\zeta = \psi_1[\sqrt{2}\,n(1+n)x_1]^{-1/(1+n)}, \tag{9}$$

then (6) becomes the ordinary differential equation

$$-\zeta = \sqrt{z}(z')^{n-2}z'' = \frac{\sqrt{z}}{n-1}\frac{d}{d\zeta}(z')^{n-1}, \tag{10}$$

where the primes denote differentiation with respect to ζ.

The boundary conditions (8) are

$$\text{when } \zeta = 0 \quad z = 0;$$
$$\text{when } \zeta = \infty \quad z = 1/2. \tag{11}$$

Equation (10) is easily integrated, if we put $z = z_0 = 1/4$ as a zeroth-order approximation (under the square root), i.e., half of its value at the outer edge of the boundary layer, as was done in [3]. We then obtain the equation

$$-\zeta = \frac{1}{2}(z')^{n-2}z'' = \frac{1}{2(n-1)}\frac{d}{d\zeta}(z')^{n-1}, \tag{12}$$

which was examined in [4, 5] for $n < 1$ in connection with the problem of unsteady motion of a non-Newtonian fluid on an infinite plate.

For n = 1 (Newtonian fluid), the solution of (12) with boundary conditions (11) has the form

$$z = \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{\zeta}\exp(-\zeta^2)\,d\zeta = \frac{1}{2}\operatorname{erf}\zeta. \tag{13}$$

Comparison of the Approximate and Exact Solutions

n	C	$z'(0)$	B_n	B_n [1]	B_n [6]	Error, %
3/1 = 3	0.4502	0.9488	0.2043	0.1941	—	5.2
2/1 = 2	0.8256	0.8256	0.3277	0.3224	—	1.6
5/3 ≅ 1.67	1.250	0.7602	0.4015	—	—	—
3/2 = 1.5	1.697	0.7200	0.4490	0.4378	—	2.6
7/5 = 1.4	2.160	0.6936	0.4828	—	—	—
4/3 ≅ 1.33	2.631	0.6742	0.5070	—	—	—
6/5 = 1.2	4.565	0.6341	0.5643	—	—	—
1	—	—	0.6709	0.6641	0.6642	1.0
4/5 = 0.8	5.782	0.4835	0.8152	—	0.8109	0.5
7/9 ≃ 0.78	5.311	0.4743	0.8350	—	—	—
3/4 = 0.75	4.851	0.4623	0.8601	—	—	—
5/7 ≃ 0.71	4.407	0.4446	0.8944	—	—	—
3/5 = 0.6	3.631	0.3935	1.019	—	1.017	0.2
1/2 = 0.5	3.404	0.3451	1.152	1.151	1.151	0.1
1/3 ≃ 0.33	3.674	0.2608	1.435	—	—	—

For $n \neq 1$, integrating (12) twice, and taking the first condition of (11) into account, we find

$$z = \int_0^{\zeta} [(n-1)(C-\zeta^2)]^{\frac{1}{n-1}} d\zeta. \quad (14)$$

The arbitrary constant C for $n < 1$ is determined from the second condition of (11), and for $n > 1$ from the condition

$$\text{when } \zeta = \zeta_\delta \; z' = 0, \; z = 1/2, \quad (15)$$

where ζ_δ is the value appropriate to the finite thickness of the boundary layer [1, 5].

The integral (14), as an integral of a binomial differential, may be expressed in finite form only for certain values of n.

In view of (7), the velocity profiles are calculated from the formula

$$u_1(\zeta) = \sqrt{2z(\zeta)}. \quad (16)$$

From (5) and (9) it follows that

$$\eta = y_1 [\sqrt{2}\, n\, (1+n)\, x_1]^{-\frac{1}{1+n}} = \int_0^{\zeta} \frac{d\zeta}{u_1(\zeta)}. \quad (17)$$

Equations (16) and (17) give the parametric relation between u_1 and η.

The calculated velocity profiles are shown in the figure. Also shown, is the curve corresponding to the exact solution of the Blasius equation [2] (dashed curve, n = 1); its difference from the approximation is less than 4%.

With the aid of (1), (3), (5), and (9) let us determine the local frictional drag of the plate

$$c_f = 2\tau_w/\rho U^2 = B_n/R_x^{1/(1+n)}, \quad (18)$$

where

$$B_n = 2\,[\sqrt{2}\, n(1+n)]^{-n/(1+n)}\,[z'(0)]^n, \; R_x = \rho U^{2-n} x^n/K. \quad (19)$$

The quantities required in calculating the velocity profiles and drag are given in the table.

The table also shows a comparison of some of the values B_n obtained in the present paper with the exact data of the authors of [1] and [6]. From the good agreement of these quantities and of the velocity profiles for n = 1 the conclusion may be drawn that the first approximation used in solving Eq. (10) is adequate for practical purposes.

NOTATION

τ—frictional stress, τ_w—the same at the wall; K, n—rheological characteristics of fluid; x—longitudinal coordinate; y—transverse coordinate; u, v—projection of velocity vector on x and y axes, respectively; U—velocity of external stream; L—characteristic length; R—Reynolds number; R_x—local Reynolds number.

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Kalinin Polytechnic Institute, Leningrad